2. OZIRANER A.S. and RUMYANTSEV V.V., Lyapunov's function method in the problem of the stability of motion with respect to some of the variables. PMM, 36, 2, 1972.
3. RUCHE N., ABETS P. and LALOIS M., Lyapunov's direct method in the theory of stability. Moscow, Mir, 1980.
4. AMANS M. and FALB P., Optimal control. Moscow, Mashinostroenie, 1968.
5. FURASOV V.D., Stability of motion, estimates and stabilization. Moscow, Nauka, 1977.
6. VOROTNIKOV V.I., Stability of motion with respect to some of the variables for some non-linear system. PMM, 43, 3, 1979.
7. MERKIN D.R., Introduction to the theory of automatic control and its application. Moscow, Nauka, 1976.
8. SYU D. and MEIER A., Current theory of automatic control and its application. Moscow, Mashinostroenie, 1972.
9. PONTRYAGIN L.S., Ordinary differential equations. Moscow, Nauka, 1970.
10. MALKIN I.G., Theory of stability of motion. Moscow, Nauka, 1966.
11. RAUSHENBAKH B.V. and TOKAR E.N., Orientation control of space craft. Moscow, Nauka, 1974.
12. ZUBOV V.I., The problem of the stability of control processes. Leningrad, Sudostroenie, 1980.
13. CHERNOUS'KO F.L., AKULENKO D.D. and SOKOLOV B.N., Oscillation control. Moscow, Nauka, 1965.
14. BELETSKII V.V., Motion of an artificial satellite with respect to the centre of mass. Moscow, Nauka, 1965.
15. GRODZOVSKII V.L., OKHOTSIMSKII D.E. et al., The mechanics of space flight. In: Mechanics the USSR over the last 50 years. Moscow, Nauka, 1, 1968.

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ON EVOLUTIONARY MOTIONS OF A PARTICLE IN A GRAVITATIONAL FIELD*

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Non-conservative, time-periodic perturbations of the Kepler problem are studied. A phase-averaged system is given, which determines the evolution in the system when there are no resonance modes. The qualitative behaviour of the solutions in the resonance zones is studied. Depending on the structure of the behaviour of the solutions, the resonances are divided into traversable, partially traversable and non-traversable. The boundedness of the set of partially traversable resonances is established, and this, in many cases, makes it possible to determine evolution in a system with resonance modes. An example is used to illustrate the method. It is shown that a constant component in the periodic function of the perturbation causes the evolutionary process to become non-unidirectional.

1. Formulation of the problem.

Consider the motion of a "particle" in a gravity field in a medium whose resistances R depend periodically on time. If r, φ are polar coordinates in the orbital plane, then the normal and tangential component of the resistance force is equal to $-mRr^{\cdot}/v$, $-mR\varphi^{\cdot}r/v$ respectively. Here m is the mass of the particle, R is the resistance per unit mass of the particle and v is the orbital velocity /1/. We write $R = \varepsilon g(r, v, \Omega t)$ where ε is a small positive parameter, the function g is at least continuous in t and periodic in Ωt with period 2π, Ω is the perturbation frequency. We also assume that g is analytic in r and $v(r)$ in the region $r \geqslant r_- > 0$. The equations of motion of the particle can be written in the form

$$r^{\cdot\cdot} - \alpha^2/r^3 + M/r^2 = -\varepsilon g r^{\cdot}/v, \quad \alpha^{\cdot} = -\varepsilon\alpha g/v \tag{1.1}$$

where $\alpha = r^2\varphi^{\cdot}$ is the kinetic moment of the particle, $M = G(m_0 + m)$, m_0 is the mass of the central body and G is the gravitational constant.

A characteristic feature of system (1.1) is the resonances, i.e. the integer-type relations connecting the perturbation frequency Ω with the characteristic frequency (with mean angular motion ω):

$$p\omega = q\Omega \tag{1.2}$$

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If we consider two (or more) mutually non-interacting particles with frequencies ω_1 and ω_2, then the resonance relations of the form (1.2) for these particles will imply that the frequencies ω_1 and ω_2 are commensurable.

Such type of commensurability is well-known in the case of the solar planetary system and of artificial satellite systems /2, 3/. One of the reasons for this phenomenon is the action of forces of resistance of the medium and tidal forces. We also note that a major part is played by the forces of resistance when a particle moves through the atmosphere.

We consider the model (1.1) in order to study the effect of the non-conservative forces of resistance of the medium on the motion of the particle, and before anything else we find out the structure of the resonance zones and investigate the possibility that the particle is arrested at the resonance. The study of system (1.1) is of some mathematical interest. A similar system with 3/2 degrees of freedom was discussed in /4, 5/. System (1.1) represents, essentially, a system with two degrees of freedom, therefore we encounter new problems in its investigation.

When $\varepsilon = 0$, system (1.1) admits of first integrals

$$r^{\cdot 2}/2 + \alpha^2/(2r^2) - M/r = h, \ \alpha = \text{const} \tag{1.3}$$

The values $h \in (-M^2/(2\alpha^2), 0)$, $h = -M^2/(2\alpha^2)$, and $h = 0$ correspond to elliptic, circular and parabolic orbits respectively. Using (1.3), we obtain the relations

$$r = a(1 - e\cos\xi), \ \theta = \omega t = \xi - e\sin\xi$$

defining the solution of the system. Here a is the major semi-axis of the Keplerian orbit, e is its eccentricity, θ is the mean anomaly and ξ is the eccentric anomaly. We replace the variables $(r, r^\cdot)$ in the region corresponding to the values $h \in [-M^2/(2\alpha^2), 0)$, $\alpha > 0$, by the action I and angle θ variables

$$I(h) = \frac{1}{2\pi}\oint r^\cdot\, dr = \frac{M}{\sqrt{-2h}} - \alpha, \quad 0 \leqslant I < \infty$$

$$\theta = \frac{\partial S(r, I)}{\partial I}, \quad S = \int_{r_*}^{r} \left[2\left(h(I) - \frac{\alpha^2}{2x^2} + \frac{M}{x}\right)\right]^{1/2} dx$$

In the new variables the unperturbed motion will be written in the form $I^\cdot = 0$, $\theta^\cdot = \omega(I, \alpha)$, $\alpha^\cdot = 0$, and we obtain the following relations for the parameters of the elliptic orbit:

$$a = -\frac{M}{2h} = \frac{(I+\alpha)^2}{M}, \quad e = \left[1 - \frac{\alpha^2}{(I+\alpha)^2}\right]^{1/2} \tag{1.4}$$
$$\omega = \frac{M^2}{(I+\alpha)^3}$$

We note that the condition $\alpha = 0, I \neq 0$ leads to $e = 1$. However, in this case the first equation of (1.1) degenerates and the motion takes place not along the parabolic trajectory, but along the trajectory for which $r \to 0$ as $t \to \infty$ (an asymptotic fall).

Further, according to /1/ we have

$$v = \left[M\left(\frac{2}{r} - \frac{1}{a}\right)\right]^{1/2} = \left[\frac{2M}{r} - \frac{M^2}{(I+\alpha)^2}\right]^{1/2} \tag{1.5}$$

Now let $\varepsilon \neq 0$. In the new variables the system (1.1) will be written in the form

$$\begin{aligned}
I^\cdot &= -\varepsilon g(r, v, \psi)\, r^\cdot r_\theta'/v + (r_\alpha' r_\theta^{\cdot\prime} - r_\theta' r_\alpha^{\cdot\prime})\, \alpha^\cdot \equiv \varepsilon F_1 \\
\alpha^\cdot &= -\varepsilon\alpha g(r, v, \psi)/v \equiv \varepsilon F_2 \\
\theta^\cdot &= \omega + \varepsilon g(r, v, \psi)\, r^\cdot r_I'/v + (r_I' r_\alpha^{\cdot\prime} - r_\alpha' r_I^{\cdot\prime})\, \alpha^\cdot \equiv \omega + \varepsilon F_3 \\
\psi^\cdot &= \Omega; \quad (F_k = F_k(I, \theta, \alpha, \psi), \ k = 1, 2, 3)
\end{aligned} \tag{1.6}$$

The system (1.6) is defined in $G \times S^1 \times S^1$, where $G = \{(I, \alpha)\colon 0 \leqslant I < \infty, \sqrt{Mr_-} < \alpha < \infty\}$. Let us consider, together with (1.6), the phase-averaged system (PAS) defined in G

$$I^\cdot = \varepsilon B_1(I, \alpha), \quad \alpha^\cdot = \varepsilon B_2(I, \alpha) \tag{1.7}$$

$$B_k = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_k(I, \theta, \alpha, \psi)\, d\theta\, d\psi$$

We assume that system (1.7) has in G at most a finite set of equilibrium states. If there were no resonance modes in system (1.6), then (1.7) would describe the evolution in the initial system. Using the last relation of (1.4), we will rewrite the condition of resonance (1.2) in the form $I + \alpha = (pM^2/(q\Omega))^{1/3}$. The condition singles out in the plane (I, α) the lines which will be called resonance lines. As we know /6/, not every point on the reference line in (1.6) has a periodic solution with the corresponding period.

Let us denote by M_{pq} the set of resonance points (I_{pq}, α_{pq}), for which such solutions exist. We show in this paper, that when there is a non-conservative perturbation $(B_{1,2} \not\equiv 0)$, the set M_{pq} will be, at most, finite. In this case the evolution in the system (1.6) will be described by motion along the trajectories of PAS, until the representative point $(I(t), \alpha(t))$ arrives in the neighbourhood of the resonance point $(I_{pq}, \alpha_{pq}) \in M_{pq}$. We further carry out a qualitative study of the behaviour of the solutions of (1.6) in the neighbourhood of the resonance points. We will show that the representative point is either arrested at the given resonance, or it passes through it and continues to move along the trajectory of PAS. If the representative point is not stopped at any resonance, then it tends, as $t \to \infty$, to the attraction set of PAS, or leaves the region G altogether. Here, as in /4/, the resonances can be divided into traversable, partially traversable and non-traversable. We will also solve the problem of the stability of the resonance modes. The case $g = \nu[\delta - (b + \sin\psi)/r]$, where δ, b are parameters, is used to illustrate the investigation.

The case $g = \nu - f(\psi)$ was studied in /7/, but in fact the work was confined to the case $f \equiv 0$ in which the particle falls asymptotically onto a neutral body (as $t \to \infty$).

The averaged system given in /7/ for the non-autonomous case does not describe the structure of the resonance zone, nor does it solve the problem of the stability of the resonance modes.

2. Auxilliary transformations of the initial system in the resonance case.

Carrying out in (1.6) the substitution

$$I = I_{pq} + \mu x_1, \quad \alpha = \alpha_{pq} + \mu x_2, \quad \theta = \Phi + q\psi/p, \quad \mu = \sqrt{\varepsilon}$$

expanding the right-hand sides of the resulting system in powers of μ and separating the terms independent of ψ ("autonomous terms"), we arrive at the system (for greater detail see /8/; here we merely note that $u_k = x_k + O(\mu)$, $\Phi = v + O(\mu^2)$)

$$\begin{aligned} &u_k^{\cdot} = \mu A_k + \mu^2 (P_{k1}u_1 + P_{k2}u_2) + \mu^3 S_k \\ &v^{\cdot} = \mu b_1 (u_1 + u_2) + \mu^2 [b_2 (u_1^2 + u_2^2) + Q] + \mu^3 S_3 \\ &\psi^{\cdot} = \Omega, \quad k = 1, 2 \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &A_k = \frac{1}{2\pi p}\int_0^{2\pi p} F_k\, d\psi, \quad Q = \frac{1}{2\pi p}\int_0^{2\pi p} F_3\, d\psi \\ &P_{kj} = \frac{1}{2\pi p}\int_0^{2\pi p} \frac{\partial F_k}{\partial y_j}\, d\psi, \quad k, j = 1, 2 \\ &F_l = F_l(I_{pq}, \alpha_{pq}, v + q\psi/p, \psi), \quad l = 1, 2, 3, \quad y_1 = I, \quad y_2 = \alpha \\ &A_k = A_k(v; I_{pq}, \alpha_{pq}); \quad P_{kj} = P_{kj}(v; I_{pq}, \alpha_{pq}) \\ &b_1 = -3M^{-4/3}(q\Omega/p)^{4/3}, \quad b_2 = 6M^{-7/3}(q\Omega/p)^{7/3} \end{aligned} \tag{2.2}$$

The functions $S_k(v, u_1, u_2, t; \mu)$ are analytic in u_1, u_2, μ for sufficiently small μ, analytic and $2\pi p/(q\Omega)$-periodic in v, continuous and $2\pi p/\Omega$-periodic in t. Carrying out the substitution $u_k = \eta_k - \mu Q/(2b_1)$ $(k = 1, 2)$ in (2.1) and neglecting terms of order $O(\mu^3)$, we obtain the system

$$\begin{aligned} &\eta_k^{\cdot} = \mu A_k(v) + \mu^2 [(P_{k1}(v) + Q_v'(v)/2)\eta_1 + (P_{k2}(v) + \\ &\quad Q_v'(v)/2)\eta_2], \quad k = 1, 2 \\ &v^{\cdot} = \mu b_1(\eta_1 + \eta_2) + \mu^2 b_2(\eta_1^2 + \eta_2^2) \end{aligned} \tag{2.3}$$

In contrast to system (1.7) (PAS), we will call system (2.3) the averaged near the resonance system (RAS).

In what follows, we will find it convenient to transform system (2.3) with help of the linear transformation $\gamma_1 = \eta_1 + \eta_2, \gamma_2 = \eta_2$, to the form

$$\begin{aligned} &v^{\cdot} = \mu b_1\gamma_1 + \mu^2 Q_1, \quad \gamma_1^{\cdot} = \mu(A_1 + A_2) + \mu^2 Q_2, \quad \gamma_2^{\cdot} = \mu A_2 + \\ &\mu^2 Q_3 \\ &Q_1 = b_2[(\gamma_1 - \gamma_2)^2 + \gamma_2^2], \quad Q_2 = (P_{11} + P_{21} + Q_v')\gamma_1 + \\ &\quad (P_{22} - P_{11} + P_{12} - P_{21})\gamma_2, \quad Q_3 = (P_{21} + \\ &\quad Q_v'/2)\gamma_1 + (P_{22} - P_{21})\gamma_2 \end{aligned} \tag{2.4}$$

3. Investigation of RAS.

First we consider the truncated system

$$v^{\cdot} = \mu b_1\gamma_1, \quad \gamma_1^{\cdot} = \mu(A_1(v) + A_2(v)), \quad \gamma_2^{\cdot} = \mu A_2(v) \tag{3.1}$$

Differentiating the first equation in t, we arrive at the phase equation

$$v^{\cdot\cdot} - \mu^2 b_1(A_1(v) + A_2(v)) = 0 \tag{3.2}$$

If the system

$$A_k(v, I, \alpha) = 0, \quad k = 1, 2, \quad I + \alpha = [M^2/(q\Omega/p)]^{1/3} \tag{3.3}$$

has a simple root v_0, I_{pq}, α_{pq}, then equilibrium states with $v = v_0$ exist for system (3.1) for these values of I and α. The equilibrium states are not isolated (they fill the straight line $\gamma_1 = 0,\ v = v_0$). Note that the phase space of system (3.1) is represented by the direct product $D \times R^1$, where $D = R^1 \times S^1$ is the phase space of system (3.2). Therefore in order to construct the phase pattern of (3.1) it is sufficient to construct the phase pattern of (3.2).

It can be shown that the smallest period of the function $A_k(v)$ in v is equal to $2\pi/p$ (see (3.7) below). Therefore, Eq.(3.2) is the pendulum equation and admits of the first integral

$$v^{\cdot 2}/2 - \mu^2 b_1 V(v) = H, \quad V = \int (A_1(v) + A_2(v))\, dv \tag{3.4}$$

which determines the phase curves. The simple equilibrium state $(\gamma_1 = 0, v = v_0)$ of Eq.(3.2) is of centre-type, provided that $b_1(A_1'(v_0) + A_2'(v_0)) < 0$, and of saddle-type if $b_1(A_1'(v_0) + A_2'(v_0)) > 0$. By virtue of (2.2) $b_1 < 0$ and hence of $A_1'(v_0) + A_2'(v_0) > 0$ is the condition for the centre, and $A_1'(v_0) + A'_2(v_0) < 0$ for the saddle. Since $A_1'(v) + A_2'(v)$, the functions $A_1(v) + A_2(v)$ take different signs in the neighbourhood zeros and the simple equilibrium states of the centre and saddle type alternate.

Let us write the function $A_k(v)$ in the form

$$A_k(v; I_{pq}, \alpha_{pq}) = A_{k*}(v; I_{pq}, \alpha_{pq}) + B_k(I_{pq}, \alpha_{pq}) \tag{3.5}$$

where B_k are the values of the functions $A_k(v)$ averaged over a period. If $B_k(I_{pq}, \alpha_{pq}) = 0$, $k = 1, 2$, then the point (I_{pq}, α_{pq}) represents the equilibrium state of PAS. The possible phase pattern of (3.2) is shown for this case in Fig.1a. We shall call such resonances non-traversable. Fig.1b shows the phase pattern for (3.2) in the case when $\max_v |A_{1*} + A_{2*}| > |B_1 + B_2| > 0$. We shall call such resonances partially traversable. When (3.2) has no equilibrium states, the initial system has no corresponding resonances modes (the converse is generally not true). The phase pattern of (3.2) for this case is shown in Fig.1c, and we shall call such resonances traversable (for more accurate definition see /8/).

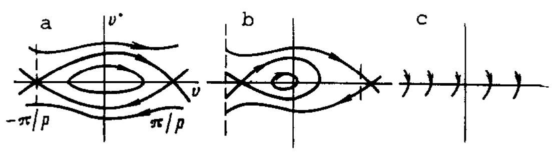

Fig.1

We note that the truncated system (3.1) does not provide a solution to the problem of the stability of resonance modes of the system (1.6) (in the class of non-conservative systems (3.1) is not structurally stable). This makes it necessary to investigate system (2.4).

Let $G_{n\rho} = \{(I, \alpha): (I - I_n)^2 + (\alpha - \alpha_n)^2 < \rho\}$, where (I_n, α_n) are the equilibrium states of PAS for which $I_n \geqslant 0$, $\alpha_n \geqslant \sqrt{Mr_-} > 0$, ρ is a sufficiently small, fixed positive number. We denote by G_* a bounded region in the plane (I, α) obtained from G by removing the neighbourhoods $G_{n\rho}$ and "infinities" $I \leqslant I_+ < \infty$, $\alpha \leqslant \alpha_+ < \infty$.

Assertion 1. System (2.4) has equilibrium states for $(I_{pq}, \alpha_{pq}) \in G_*$ for not more than a finite set $\{p, q\}$.

To prove this assertion it is sufficient to show that system (3.2) has no equilibrium states when p and q are sufficiently large. Expanding the functions F_k in dual Fourier series

$$F_k(I, \alpha, \theta, \psi) = \sum_{m, s=-\infty}^{\infty} F_{kms}(I, \alpha) \exp[i(m\theta + s\psi)]$$

we obtain from (2.2)

$$A_k(v; I_{pq}, \alpha_{pq}) = \frac{1}{2\pi p} \sum_{m, s=-\infty}^{\infty} F_{kms}(I_{pq}, \alpha_{pq}) \int_0^{2\pi p} \exp\{i[m(v + q\psi/p) + s\psi]\}\, d\psi \tag{3.6}$$

The integral in (3.6) is different from zero when $m = np$, $s = -nq$, n is an integer, therefore we have

$$A_k(v; I_{pq}, \alpha_{pq}) = \sum_{n=-\infty}^{\infty} F_{k, np, -nq}(I_{pq}, \alpha_{pq}) \exp(inpv) \tag{3.7}$$

The following expansions /1/ hold for the unperturbed solution $r(\theta)$:

$$r = a\left[1 + \frac{e^2}{2} - e\sum_{m=1}^{\infty}\frac{1}{m}\left(J_{m-1}(me) - J_{m+1}(me)\right)\cos m\theta\right] \tag{3.8}$$

$$\frac{a}{r} = 1 + 2\sum_{m=1}^{\infty} J_m(me)\cos m\theta$$

where $J_s(me)$ is a Bessel function of order s. The following estimate /9/ holds for large s:

$$J_s(se) \sim \{\exp[s(\operatorname{th}\beta - \beta)]\,(2\pi s \operatorname{th}\beta)^{-1/2},\ \operatorname{ch}\beta = e^{-1} > 1 \tag{3.9}$$

Further, the expansion

$$g = \sum_{n=-\infty}^{\infty} g_n(r, \nu)\exp(in\psi)$$

also holds. Then, from (1.6), (3.8), (3.9) and the analyticity of g in r and ν in the region in question, and from the continuity in t, we obtain the estimate

$$F_{k,\,np,\,-nq} \sim \exp[-C(e)\,|n\,|\,p]\,(|\,n\,|\,q)^{-1} \quad \text{when} \quad \frac{p}{q} \to \frac{\Omega}{\omega_*} > \frac{\Omega}{\omega_0}, \quad n \neq 0$$

$C(e) > 0$ for $e < 1$, $\omega_0 = M^2/\alpha^3$, $\omega_* = \omega(I_*, \alpha_*)$, $(I_*, \alpha_*) \in G_*$.

This, together with the relations $F_{k,0,0} \equiv B_k\ (k = 1, 2)$, yields the required assertion.

Notes. 1°. If $\omega_* = \omega_0$, then we have $I \to 0, e \to 0$ as $(p/q) \to (\Omega/\omega_*)$, and therefore $r \to r_0 = \alpha^2/M$, $r_\theta' \to 0$. Then $B_1 \to 0$ and $B_2 \not\equiv 0$, if

$$\int_0^{2\pi} g(r_0, \nu(r_0), \psi)\,\nu^{-1}\,d\psi \not\equiv 0$$

otherwise the PAS has an infinite set of equilibrium states which contradicts the assumption made above. If on the other hand the equation

$$\int_0^{2\pi} g(r_0, \nu(r_0), \psi)\,\nu^{-1}\,d\psi = 0$$

has isolated roots $\alpha = \alpha_j$, then $I = 0$ represents the coordinate of the equilibrium state of PAS, i.e. $(I = 0, \alpha_j) \equiv G_*$.

2°. Assertion 1 enables us to study system (2.4) in the neighbourhood of individual resonance points (I_{pq}, α_{pq}) with the global investigation in G_*, and thus solve the problem of evolution in system (1.6).

Let us consider the problem of the stability of the equilibrium states of system (2.4). If $v = v_0$ is the coordinate of the equilibrium state of the truncated system (3.1), then system (2.4) has an isolated equilibrium state $(v = v_0, \eta_1 = \eta_2 = 0)$ provided that $A_0'(v_0) + A_2'(v_0) \neq 0$. The characteristic equation for this equilibrium state has the form

$$\begin{gathered} -\lambda^3 + \mu^2 a_2 \lambda^2 + \mu^2 (a_1 + O(\mu^2))\,\lambda + \mu^4 a_0 = 0 \\ a_0 = b_1 [A_1'(P_{21} - P_{22}) + A_2'(P_{12} - P_{11})]_{v=v_0} \\ a_1 = b_1 (A_1' + A_2')\,|_{v=v_0}, \quad a_2 = (P_{11} + P_{22} + Q_v')\,|_{v=v_0} \end{gathered} \tag{3.10}$$

According to /6/ we have the following asymptotic expression for the roots of (3.10):

$$\lambda_{1,2} = \pm\sqrt{a_1}\,\mu - (a_0 - a_1 a_2)(2a_1)^{-1}\mu^2, \quad \lambda_3 = -a_0 a_1^{-1}\mu^2$$

From this it follows, that when the conditions

$$a_1 < 0, \quad a_0 < 0, \quad a_2 < a_0/a_1 \tag{3.11}$$

hold, the equilibrium state $(v_0, 0, 0)$ of system (2.4) is stable (a generalized node). Thus, for the centre-type equilibrium state of system (3.2) under the condition $a_0 < 0$, $a_2 < a_0/a_1$ we have the corresponding stable equilibrium state of the RAS. If even a single condition of (3.11) is violated, then the equilibrium state will not be asymptotically stable (when the inequality is strict, the state is unstable).

Let us introduce the function

$$\sigma(v; I_{pq}, \alpha_{pq}) = P_{11}(v; I_{pq}, \alpha_{pq}) + P_{22}(v; I_{pq}, \alpha_{pq}) + Q_v'(v; I_{pq}, \alpha_{pq})$$

defining the divergence of the vector field of RAS. Clearly, $\sigma(v_0) = a_2$. Using (2.2) and the fact that the Jacobian of the transformation $(r, r^{\cdot}) \to (I, \theta)$ is equal to unity, we obtain

$$\sigma(v) = -\frac{1}{\pi p}\int_0^{2\pi p} g\nu^{-1}\,d\psi \tag{3.12}$$

From (3.12), (2.2), (1.6) it follows that

$$\sigma(v) = 2A_2(v)\,\alpha^{-1} \tag{3.13}$$

and this proves

Assertion 2. If system (2.4) has a single equilibrium state $(v_0, 0, 0)$, then $\sigma(v)$ is a sign-alternating function and $\sigma(v_0) = a_2 = 0$.

In order to establish the topology of RAS in the case of sign-alternating $\sigma(v)$, we pass, in the truncated system (3.1) in the region $D_0 \times R^1$, where D_0 is the area of the plane $(v, v^{\cdot})$ filled with the closed phase curves of (3.2), from the variables (v, γ_1, γ_2) to the variables (J, β, γ_2), where J is the action and β the angle. When $B_k = 0\ (k = 1, 2)$ then the action J-angle β variables can be used not only in the region of oscillatory motions of (3.2), but also in the region of rotational motions.

In the new variables system (2.4) takes the form (a prime denotes a derivative with respect to $\tau = \mu t$)

$$\begin{gathered} J' = \mu R_1, \quad \beta' = \omega_p(J) + \mu R_2, \quad \gamma_2' = A_2(v(J, \beta)) + \mu R_3 \\ \omega_p = dH/dJ, \quad R_1 = Q_2 v_\beta' - Q_1\gamma_{1\beta}', \quad R_2 = -Q_2 v_J' + Q_1\gamma_{1J}' \\ R_3 \equiv Q_3 \quad (R_k = R_k(J, \beta, \eta_2), \quad k = 1, 2, 3) \end{gathered} \tag{3.14}$$

Carrying out the substitution

$$\gamma = \gamma_2 - \omega_p^{-1}\int A_2(v(J, \beta))\,d\beta$$

we transform system (3.14) to the form

$$\begin{gathered} J' = \mu R_1, \quad \gamma' = \mu\,[R_3 - A_2(v(J, \beta))\,R_2\omega_p^{-1}] \\ \beta' = \omega_p + \mu R_2 \end{gathered} \tag{3.15}$$

The right-hand sides of system (3.15) are 2π-periodic in β. Averaging (3.15) over β, we arrive at the system

$$\begin{gathered} \bar{J}' = \mu C_1(\bar{J}, \bar{\gamma}), \quad C_1 = \frac{1}{2\pi}\int_0^{2\pi} (R_3 - A_2 R_2 \omega_p^{-1})\,d\beta \\ \bar{\gamma}' = \mu C_2(\bar{J}, \bar{\gamma}), \quad C_2 = \frac{1}{2\pi}\int_0^{2\pi} R_1\,d\beta \end{gathered} \tag{3.16}$$

If system (3.16) has a simple, stable equilibrium state $(J_0, \bar{\gamma}_0)$ in the domain of admissible values of $J, \bar{\gamma}$, then, as we know (e.g. /10/), we have in RAS the corresponding stable limit cycle provided that $J_0 \neq 0$, or a stable equilibrium state if $J_0 = 0$. If on the other hand (3.16) has a stable coarse limit cylce of frequency l, then we have in RAS a corresponding stable, two-dimensional invariant torus T^2 which divides the phase space of RAS. The solutions on T_2 are two-frequency solutions with the frequencies ω_p and l. The solution on the cycle, as well as solutions on the torus, all have long periods, otherwise $\tau = \mu t$. We find that stable periodic solutions in (1.6) correspond to simple, stable equilibrium states of RAS, two-dimensional stable invariant tori correspond to the stable coarse limit cycles, and the stable three-dimensional tori to the stable two-dimensional tori.

4. **Example.** Let us put $g = \nu(\delta - f(\psi)/r)$, where δ is a parameter.

1°. Let $f = \sin\psi$. Then according to (1.6) we have

$$\begin{gathered} F_1 = -\left(\delta - \frac{\sin\psi}{r}\right) r^{\cdot} r_\theta' - \alpha\left(\delta - \frac{\sin\psi}{r}\right)\left(\frac{\alpha}{\omega r^2} - 1\right), \\ F_2 = -\alpha\left(\delta - \frac{\sin\psi}{r}\right) \end{gathered} \tag{4.1}$$

First we calculate the right-hand sides of PAS. Substituting (4.1) into the last relation of (1.7) we obtain

$$B_1 = -\delta I, \; B_2 = -\delta\alpha \tag{4.2}$$

In this case the system (1.7) (PAS) will have a unique equilibrium state at the origin of coordinate of the subcritical node type (stable at $\delta > 0$). Therefore, when there are no resonances, the particle falls asymptotically when $\delta > 0$ on to the neutral body.

Let us now compute the right-hand sides of the truncated system (3.1). We have, in accordance with (2.2),

$$\begin{gathered} A_1 \equiv B_1 + A_{1*}(v), \quad A_2 = B_2 + A_{2*}(v) \\ A_{1*} = \frac{1}{2\pi p}\int_0^{2\pi p}\left[r^{\cdot} r_\theta' \frac{\sin\psi}{r} + \alpha\frac{\sin\psi}{r}\left(\frac{\alpha}{\omega r^2} - 1\right)\right] d\psi \\ A_{2*} = \frac{\alpha}{2\pi p}\int_0^{2\pi p} \frac{\sin\psi}{r}\,d\psi \end{gathered} \tag{4.3}$$

From (3.8) we obtain

$$r_\theta' = ae\sum_{n=1}^{\infty}(J_{n-1}(ne) - J_{n+1}(ne))\sin n\theta \tag{4.4}$$

Substituting (4.4) and the second equation of (3.8) into (4.3) and taking into account the fact that $r^{\cdot} = \omega r_\theta'$, we obtain

$$\begin{aligned}
&A_{1*} = A_{1p}\sin pv, \quad A_{2*} = A_{2p}\sin pv \\
&A_{1p} = \frac{\omega a e^2}{4}\left[\left(J_{p/2-1}\left(\frac{pe}{2}\right) - J_{p/2+1}\left(\frac{pe}{2}\right)\right)^2 - \right.\\
&\quad \left. 2J_p(pe)(J_{p-1}(pe) - J_{p+1}(pe))^2\right] - \frac{3\alpha^2}{\omega a^3}\left(J_p(pe) + J_p^3(pe) + \right.\\
&\quad \left. J_{p/2}^2\left(\frac{pe}{2}\right)\right) + \frac{\alpha}{a}J_p(pe), \quad p - \text{even} \quad q = 1 \\
&A_{1p} = -\frac{\omega a e^2}{2}J_p(pe)(J_{p-1}(pe) - J_{p+1}(pe))^2 - \\
&\quad \frac{3\alpha^3}{\omega a^3}(J_p(pe) + J_p^3(pe)) + \frac{\alpha}{a}J_p(pe), \quad p - \quad \text{odd}, \quad q = 1 \\
&A_{2p} = -\alpha a^{-1}J_p(pe), \; q = 1
\end{aligned} \tag{4.5}$$

When $q > 1$ we have $A_{1p} = 0, A_{2p} = 0$.

To solve the problem of the existence of equilibrium states of system (3.1), and hence of (2.4), we consider system (3.3). To be specific we will assume that p is odd. Eliminating $\sin pv$ from the equation $A_1(v) = 0$, with help of the equation $A_2 = 0$, we arrive at the equation

$$e^2(J_{p-1}(pe) - J_{p+1}(pe))^2 + 6(1 - e^2)(1 + J_p^2(pe)) - 2 = 0 \tag{4.6}$$

Thus system (2.4) has equilibrium states for those odd p for which the transcendental Eq.(4.6) has a simple root $e = e_p \in [0, 1)$, such that

$$|\delta d_p^3/(MJ_p)| < 1; \quad d_p = [M^2/(\Omega/p)]^{1/3} = I + \alpha \tag{4.7}$$

Using the expansion

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(n+k)!}\left(\frac{z}{2}\right)^{2k}$$

we can show that Eq.(4.6) has no roots with sufficiently small values of e_p (a comet-shaped particle). For example, for $p = 1$ we have $e_1^2 \approx 0{,}77$.

When condition (4.7) holds, the equations $B_k + A_{k*}(v) = 0$ $(k = 1, 2)$ have two real roots within the period $\pi/p \leqslant v < \pi/p$.

Let us denote by v_0 the root for which $\cos pv_0 < 0$. With e_p known, we use the second formula of (1.4) to find $\alpha = \alpha_{p1} = d_p\sqrt{1 - e_p^2}$. Finally, from the relation $I + \alpha = d_p$ we obtain $I = I_{p1} = \alpha_p - \alpha_{p1}$. In this manner we determine the resonance point (I_{p1}, α_{p1}), in whose $\sqrt{\varepsilon}$-neighbourhood system (1.6) has resonance modes. In other words, we have determined the parameters of the elliptic generating phase curve of the equation $r^{\cdot\cdot} - \alpha^2/r^3 + M/r^2 = 0$.

In the course of solving the problem of the stability of the equilibrium state $(v_0, 0, 0)$ of system (2.4), we must determine the signs of the quantities a_1, a_0 (according to assertion 2 $a_2 = 0$). This can be done using the formulas for a_0, a_1 in (3.10), (4.5), (4.7), (2.2). Thus for small p we have $a_1 < 0$ when $\cos pv_0 < 0$. Further, it can be shown that $a_0 = \delta E$, where $E = E(p)$ is independent of δ. Therefore, when $E \neq 0$, we can attain the condition of stability $a_0 < 0$ by choosing the sign of the parameter δ. The calculation of E is cumbersome and is therefore omitted.

2°. Let us consider the case $f = b + \sin\psi$, where b is a constant term. In this case the PAS has the form

$$\begin{aligned}
&I^{\cdot} = \varepsilon[-\delta I + bM(I/[\alpha(I + \alpha)] + 1/\alpha - \alpha/(I + \alpha)^2] \\
&\alpha^{\cdot} = \varepsilon[-\delta\alpha + bM(\alpha/(I + \alpha)^3)]
\end{aligned} \tag{4.8}$$

Clearly, the functions $A_{1*}(v), A_{2*}(v)$ remain as before (see (4.5)). System (4.8) in the region G has a unique unstable equilibrium state $(I = 0, \alpha = \sqrt{bM/\delta})$ of the saddle type. A circular orbit with $r_0 = b/\delta$ corresponds to this state of equilibrium.

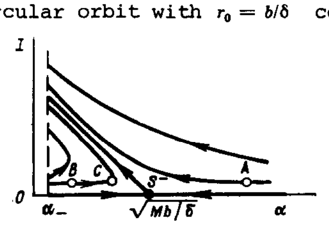

Fig.2

Fig.2 shows a possible phase pattern of system (4.8) with $\delta > 0, b > 0$. The appearance of a saddle-type equilibrium state leads to the fact that the evolution has a different character on each side of the unstable separatrix S^-. If the particle was situated at the initial instant at the point corresponding to point A (Fig.2), then its further motion, provided that there are no resonance modes, follows that corresponding phase curve in the direction of the arrow. Moreover, decreasing the value of α (at constant I) leads, in accordance with (1.4), to reduction in the size of the orbit, while increasing the value of I (with α kept constant) leads to an increase in the eccentricity e. The evolution is completed when the article arrives at the dashed line (Fig.2) and then falls onto the neutral body. If on the other hand at the initial instant the particle has parameters corresponding to the point B (i.e. it lies in the orbit situated nearer to the central body), then the orbit will increase (α will increase) up to some instant corresponding to point C in Fig.2. From then on the evolution will be the same as in the previous case.

Thus when the function $f(\psi)$ has a constant term, we have an interesting effect from the point of view of the evolution of the motion. The evolutionary process may develop in more than one direction.

REFERENCES

1. SMART U.M., Celestial Mechanics. Moscow, Mir, 1965.
2. MOLCHANOV A.M., Resonances in multifrequency oscillations. Dokl. AS SSSR, 168, 2, 1966.
3. GREBENIKOV E.A. and RYABOV YU.A., Resonances and Small Denominators in Celestial Mechanics. Moscow, Nauka, 1978.
4. MOROZOV A.D. and SHIL'NIKOV L.P., On non-conservative periodic systems close to two-dimensional Hamiltonian systems. PMM 47, 3, 1983.
5. MOROZOV A.D., Systems close to non-linear integrable systems. Gor'kii, Izd-e Gor'l. un-ta, 1983.
6. VOLOSOV V.M. and MORGUNOV B.I., Method of Averaging in the Theory of Non-linear Oscillatory Systems. Moscow, Izd-vo MGU, 1971.
7. CHESTER W., A general theory of resonance between weakly coupled oscillatory systems. Pt II. Non-conservative systems. J. Inst. Math. and Appl. 26, 2, 1980.
8. MOROZOV A.D., On the study of resonances in a system of non-linear weakly coupled oscillators. In: Methods of the Qualitative Theory of Differential Equations. Gor'kii, Izd-e Gor'k, un-ta, 1984.
9. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products. Moscow, Nauka, 1971.
10. ZHEVAKIN S.A., On determining the limit cycles in systems close to certain non-linear systems. PMM 15, 2, 1951.

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